# The 1st main theorem of Complements

**Eventual Goal:** BAB, $\varepsilon$-lc Fanos of dim $d$ form a bounded family. —Birkar

On the way:

$\Phi(\mathcal{R}) := \left\{ 1 - \frac{r}{\ell} : r \in \mathcal{R}, \ell \in \mathbb{N} \right\}$, $\mathcal{R} \subset [0,1]$

$d \in \mathbb{N}$, $\mathcal{R}$ finite set of rationals. Exists $n = n(d, \mathcal{R}) \in \mathbb{N}$ such that

**Thm (Bddness of Global Complements)**

For $(X,B)$ a projective pair s.t.

1. $(X,B)$ l.c. of dim $d$
2. $\mathrm{coeffs}(B) \subset \Phi(\mathcal{R})$
3. $X$ Fano type
4. $-(K_X + B)$ nef

$(X,B)$ has a monotone $n$-complement.

Thm (Bddness of Local Complements)

For $(X,B) \to Z$ a projective contraction s.t.

① $(X,B)$ l.c. of dim $d$, $\dim(Z) > 0$ ③ $X$ Fano type $/Z$

② $\mathrm{coeff}(B) \subset \Phi(\mathcal{R})$ ④ $-(K_X+B)$ nef $/Z$

$(X,B)$ has a monotone $n$-complement over any $\sigma \in Z$

**Definition:** Let $(X,\Delta)$ be a log pair &

$X \longrightarrow Z$ be a projective contraction.

Let $N$ be a positive integer & $z \in Z$ a closed point.

We say that $B \geq 0$ on $X$ is a $N$-complement over $z \in Z$.

if the following conditions are satisfied:

i) $(X,B)$ is log canonical over a neighborhood of $z \in Z$.

ii) $N(K_X+B) \sim_Z 0$ after possibly shrinking around $z \in Z$.

iii) $NB \geq N\lfloor \Delta \rfloor + \lfloor (N+1)\Delta \rfloor$ ← Diophantine approx.

If $NB \geq N\Delta$, then we say it is a monotone $N$-comp.

$$\mathcal{S} = \left\{ 1-\frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$$

**Rmk 1:** $Z = pt$, $\Delta = 0$, $N$-complement, is an element of $|-NK_X|$ with nice sing.

**Rmk 2:** $X \longrightarrow Z$ identity and $x \in X$.

A $N$-complement is the structure of a lc sing with prescribed index

Birkar proves this by showing

$$\boxed{\text{Global}_{d-1} + \text{local}_{d-1} \Rightarrow \text{local}_d}$$

20 years prior...

Prokhorov + Shokurov prove this for $(X/Z \ni o, B)$ klt and $-(K_X+B)$ nef and big $/Z$, and $\Phi = \mathfrak{S}$

- These complements can be taken non-klt (~ lower index, helpful for induction)
- Index $\leq$ largest minimal complementary index occurring for Fano types of dim $d-1$

(or $d-2$ for $(X/Z \ni o, D)$ non-exc.)

## Definition

If $(X/Z \ni z, \Delta)$ has a $\mathbb{Q}$-complement over a neighborhood of $z$, it is exceptional if for any $\mathbb{Q}$-complement $K+\Delta^+$ of $K+\Delta$, there is $\leq 1$ prime exc. divisor $E$ of $k(X)$ with $a(E, \Delta^+) = -1$.

## Strategy

Construct a special blow up with an irreducible exceptional divisor $S$ that we may apply our inductive hypothesis and lift an $n$-complement from.

# plt blow ups

For $(X, \Delta)$ a log pair and

$$g : Y \to X$$

a blow up with one irreducible exc. divisor $S \subset Y$, $g : (Y \supset S) \to X$ is a plt blow up if:

- $K_Y + \Delta_Y + S$ plt

and

- anti-ample over $X$

# Proposition (Constructing plt blow ups)

Let $(X, \Delta + \Delta^\circ)$ be $\mathbb{Q}$-factorial with $\Delta, \Delta^\circ \geq 0$, $K + \Delta + \Delta^\circ$ l.c. but not klt, and $K + \Delta$ klt, then:

There is a plt blow up $g: (Y \supset S) \to X$ with

- $K_Y + \Delta_Y + S + \Delta^\circ_Y = g^*(K + \Delta + \Delta^\circ)$ l.c.
- $K_Y + \Delta_Y + S + (1-\varepsilon)\Delta^\circ_Y$ plt and anti-ample $/X$ for any $\varepsilon > 0$
- $Y$ $\mathbb{Q}$-factorial and $\rho(Y/X) = 1$

Called an <u>inductive blow up</u>.

proof

Let $h: V \to X$ be a log terminal modification with

$$h^*(K+\Delta+\Delta^\circ) = K_V + \Delta_V + \Delta_V^\circ + E$$

$E \neq 0$ reduced.

Write

$$h^*(K+\Delta) = K_V + \Delta_V + \sum \alpha_i E_i, \quad \alpha_i < 1 \quad \text{(klt)}$$

$$\Longrightarrow h^*\Delta^\circ = \Delta_V^\circ + \sum (1-\alpha_i) E_i$$

So, $K_V + \Delta_V + E \equiv_X -\Delta_V^\circ \equiv_X \underline{\sum (1-\alpha_i) E_i}$

cannot be nef$/X$.

Run a $(K_V+\Delta_V+E)$-MMP over $X$

$\rightsquigarrow$ get a birational contraction

last step of MMP $\rightsquigarrow$ $g: Y \to X$ satisfying above.

- $K_Y+\Delta_Y+S+\Delta_Y^\circ = g^*(K+\Delta+\Delta^\circ)$ l.c.

- $K_Y+\Delta_Y+S+(1-\varepsilon)\Delta_Y^\circ \equiv_X -\varepsilon\Delta_Y^\circ$ plt

and anti-ample over $X$ for $\varepsilon > 0$.

- $Y$ $\mathbb{Q}$-factorial and $\rho(Y/X)=1$.

Lemma (weak Fano type $\Rightarrow$ Fano type)

Let $(X/Z, D)$ klt, $-(K_X+D)$ nef/big over $X$.

There exists an effective $\mathbb{Q}$-divisor $D^{\sim}$ with $K_X+D+D^{\sim}$ klt and anti-ample over $X$.

proof

Take $A$ ample over $Z$ on $X$ and $n \gg 0$ so that $|n(K_X+D)-A| \neq \emptyset$ (Kodaira's lemma, $-(K_X+D)$ big)

Take $D'$ general inside $\Rightarrow$ $-n(K_X+D)-D' \equiv_X A$

($\Rightarrow X$ Mori Dream Space)

Let $D^{\sim}$ be $D'/n$ $\square$

# Back to Main Proof

Replace $X$ with $\mathbb{Q}$-factorialization

$\rightsquigarrow$ no assumption change

$f: X \to Z$

Take $D^{\nabla}$ as in Lemma

and $D^{\triangledown} := D^{\nabla} + c f^{*} H$ for $0 \in H \subset Z$ effective,

Cartier, and $c = \mathrm{lct}(X, D + D^{\nabla}; f^{*} H)$, max $c \in \mathbb{R}$

with $K + D^{\triangledown}$ l.c.

$K + D + D^{\triangledown}$ is anti-ample over $Z$, l.c.,

and not klt.

$K + D + D^{\triangledown}$ is plt or not plt

A B

(B) $g : (\hat{X} \supset S) \to X$ inductive blow up

of $(X, D+D^\circ)$.

where $g^*(K+D+D^\circ) = K_{\hat{X}} + \Delta + S + \hat{D}^\circ$

$g^*(K+D) = K_{\hat{X}} + \Delta + aS$ $\rightsquigarrow$ only a sub-boundary

for $\Delta := g_*^{-1} D$, $\hat{D}^\circ := g_*^{-1} D^\circ$, $a < 1$ (dlt)

(A) $\hat{X} := X$, $g := id$, $S = \lfloor D + D^\circ \rfloor$

$S$ connected by connectedness lemma

(since $K + D + D^\circ$ anti big/nef over $Z$)

and $S$ normal since $(X, D+D^\circ)$ plt $\Rightarrow$ dlt.

$\rightsquigarrow$ $S$ irreducible.

In both cases,

$K+\Delta+S+\hat{D}^{\triangledown}$ l.c. and not klt,

$K+\Delta+aS$ sub-klt

Both anti-nef/big over $Z$.

Lemma

We can increase $\Delta+aS$ to become effective while retaining these properties

$\longrightarrow$ call this boundary $M \leq \Delta+S+(1-\delta_0)\hat{D}^{\triangledown}$

Suffices to produce an n-complement on $(\hat{X}/Z \ni o, M)$ $\longrightarrow$ having an n-complement preserved under pushforward by birational contraction.

proof Define $M$ by $K_{\hat{X}}+M = g^*(K+D+(1-\delta_0)D^{\triangledown})$ for $0<\delta_0 \ll 1$.

$\Longrightarrow \overline{NE}(\hat{X}/Z)$ polyhedral.

$\hat{D}^\lambda := (1-\lambda)\hat{D}^\triangledown$ → plt → anti-ample$/Z$

$\underline{K+D+S+\hat{D}^\lambda}$ anti-ample over $Z$

For $R_g$ = fibers of $g$, $R_g \cdot (K_{\hat{X}}+\Delta+S+\hat{D}^\triangledown) = 0$

$R \cdot (\underline{K_{\hat{X}}+\Delta+S+\hat{D}^\triangledown}) < 0$, $R \neq R_g$

($K_X+\Delta+D^\triangledown$ anti-ample)

$D^\triangledown \equiv_X -(1-a)S$ positive on $R_g$

$\Longrightarrow (K_{\hat{X}}+\Delta+S+(1-\lambda)\hat{D}^\triangledown)\cdot R < 0$ for all $R$

## Lemma

We can choose a $B$ on $\hat{X}$ such that

- $K+\Delta+S+B \equiv_Z 0$ plt
- Components generate $N^1(\hat{X}/Z)$

~~proof~~

Take $B := \hat{D}^\lambda + \frac{1}{n}(F+\sum_i F_i)$ with

$$F \in \left| -n\left(K_{\hat{X}} + \Delta + S + \hat{D}^{\lambda}\right) - \sum F_i \right|$$

(basepoint free to preserve plt)

$F_i$ prime generating $N^1(\hat{X}/Z)$.

## Bank

- $g : (\hat{X} \ni S) \to X$ inductive blow up
- Boundary $M$ on $\hat{X}$ with $K_{\hat{X}} + M$ plt, anti-nef / big / $Z$
- $M \le \Delta + S + (1-\varepsilon)B$, $0 < \varepsilon \ll 1$

$\to$ a numerical complement $\Delta + S + B$ of $M$ over $Z$ with $K_{\hat{X}} + \Delta + S + B$ plt and components $B$ generating $N^1(\hat{X}/Z)$.

Run a $\big(K+\Delta+S+(1+\varepsilon)B\big)$-MMP over $Z$

$$\underbrace{K+\Delta+S+(1+\varepsilon)B}_{\equiv_Z \varepsilon B}$$

$$\begin{tikzcd}
 & \hat{X} \arrow[dl, "g"'] \arrow[dr, dashed] & \\
X \arrow[dr, "f"'] & & \overline{X} \arrow[dl, "q"] \\
 & Z &
\end{tikzcd}$$

$\varepsilon B$-MMP

$\Rightarrow$ $R\cdot B<0$ and $R\cdot(K+\Delta+S)>0$ on all extremal $R$

$\overline{\square} :=$ proper transform of $\square$ in $\overline{X}$

$K+\Delta+S+B\equiv_Z 0$

$\rightsquigarrow$ all divisorial contractions contract a component of $B$.

$\Rightarrow$ $S$ not contracted

$\varepsilon\overline{B}\equiv_Z -(K+\overline{\Delta}+\overline{S})$ nef$/Z$

## Lemma

Can arrange this MMP to preserve the existence of such an $M$, so that $\bar{X}$ is Fano type over $Z$, $K+\bar{\Delta}+\bar{S}$ is plt, and anti-nef/big/$Z$.

Suffices to produce an $n$-complement on $(\bar{X}/Z \ni o, \bar{\Delta}+\bar{S})$

$\rightsquigarrow$ $n$-complements can be pulled back via $(K+\Delta+S)$-positive divisorial contractions

Remark
$-(K+\bar{\Delta}+\bar{S})$ anti lin / nef / $Z$

Basepoint free thm $\Rightarrow -(K+\bar{\Delta}+\bar{S})$ semiample

If not ample, get a birational contraction $\phi : \bar{X} \to X'$ over $Z$ with $\mathrm{exc}(\phi) \subset \mathrm{Supp}(\bar{B})$

For any curve $C$ in a fiber,

$$C \cdot \bar{B} = 0$$

$\Rightarrow C \cdot \bar{B}_i < 0$ for some component $\bar{B}_i$ of $B$, since they generate $N^1(\bar{X}/Z)$.

Thus, $-\left(K_{\bar{S}} + \mathrm{Diff}_{\bar{S}}(\bar{\Delta})\right) = -(K + \bar{\Delta} + \bar{S})|_{\bar{S}}$

big / nef / $q(\bar{S})$

$\mathrm{exc}(\phi) \subset \mathrm{Supp}(\bar{B})$
$\not\subset$
$\mathrm{Supp}(\bar{S})$

## Prop

We may extend an $n$-complement of $(\bar{S}/q(\bar{S}) \ni o, \mathrm{Diff}_{\bar{S}}(\bar{\Delta}))$ to one of $(\bar{X}/Z \ni o, \bar{\Delta} + \bar{S})$.

## sketch

Take a log res

$$h : Y \to \bar{X}$$

and write $K_Y + S_Y + A = h^*(K_{\bar{X}} + \bar{\Delta} + \bar{S})$

$\rightsquigarrow$ gives birational contraction

$$h_S : S_Y \to \bar{S}$$

$$K_{S_Y} + \mathrm{Diff}_{S_Y}(A) = h_S^*\left(K_{\bar{S}} + \mathrm{Diff}_{\bar{S}}(\bar{A})\right)$$

$= A|_{S_Y}$, $S_Y$ smooth

$\rightsquigarrow$ get an $n$-complement

$$K_{S_Y} + \mathrm{Diff}_{S_Y}(A)^+$$

Get $\Theta \in \left| -nK_{S_Y} - \lfloor (n+1)\mathrm{Diff}_{S_Y}(A) \rfloor \right|$

s.t. $n\mathrm{Diff}_{S_Y}(A)^+ = \lfloor (n+1)\mathrm{Diff}_{S_Y}(A) \rfloor + \Theta$

Kawamata-Viehweg $H^1(Y, -nK_Y-(n+1)S_Y-\lfloor (n+1)A \rfloor)) = 0$

$\Rightarrow$

$H^0(Y, \mathcal{O}_Y(-nK_Y-nS_Y-\lfloor (n+1)A \rfloor)))$

$\twoheadrightarrow H^0(S_Y, \mathcal{O}_{S_Y}(-nK_{S_Y}-\lfloor (n+1)A \rfloor|_{S_Y}))$

Get $\Xi \in |-nK_Y-nS_Y-\lfloor (n+1)A \rfloor|$

s.t. $\Xi|_{S_Y} = \Theta$

Defining $A^+ := \frac{1}{n}(\lfloor (n+1)A \rfloor + \Xi)$

$\rightsquigarrow n(K_Y+S_Y+A^+) \sim_{\mathbb{Z}} 0$

and $n(K_Y+S_Y+A^+)|_{S_Y} = K_{S_Y} + \mathrm{Diff}_{S_Y}(A)^+$.

Set $\Delta^+ := h_* A^+$

$$\rightsquigarrow n\left(K_{\bar{X}} + \bar{S} + \bar{\Delta}^{+}\right) \sim_{\mathbb{Z}} 0$$

Can show $K_{\bar{X}} + \bar{S} + \bar{\Delta}^{+}$ l.c. using inversion of adjunction and that inequality is satisfied.